# Exotic baryons in two-dimensional QCD and the generalized sine-Gordon solitons

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ABSTRACT: The solitons and kinks of the SU(3) generalized sine-Gordon model (GSG) are shown to describe the baryonic spectrum of two-dimensional quantum chromodynamics (QCD$_2$). The GSG model arises in the low-energy effective action of bosonized QCD$_2$ with unequal quark mass parameters. The GSG potential for $N_f = 3$ flavors resembles the potential of the effective chiral lagrangian proposed by Witten to describe low-energy behavior of four dimensional QCD. Among the attractive features of the GSG model are the variety of soliton and kink type solutions for QCD$_2$ unequal quark mass parameters [JHEP(0701)(2007)(027)]. Exotic baryons in QCD$_2$ [J. Ellis et al JHEP0508(2005)081] are discussed in the context of the GSG model. Various semi-classical computations are performed improving the results of this reference and clarifying the role of unequal quark masses. The remarkable double sine Gordon model also arises as a reduced GSG model bearing a kink($K$) type solution describing a multi-baryon; so, the description of some resonances in QCD$_2$ may take advantage of the properties of the $K\bar{K}$ system.

KEYWORDS: Nonperturbative Effects, Field Theories in Lower Dimensions, Strong Coupling Expansion, QCD.

# Contents

## 1. Introduction

Quantum Chromodynamics in two-dimensions (QCD$_2$) (see e.g. [1]) has long been considered a useful theoretical laboratory for understanding non-perturbative strong-interaction problems such as confinement [2], the large-$N_c$ expansion [3], baryon structure [4] and, more recently, the chiral-soliton picture for normal and exotic baryons [5]. Even though there are various differences between QCD$_4$ and QCD$_2$, this theory may provide interesting insights into the physical four-dimensional world. In two dimensions, an exact and

complete bosonic description exists and in the strong-coupling limit one can eliminate the color degrees of freedom entirely, thus getting an effective action expressed in terms of flavor degrees of freedom only. In this way various aspects have been studied, such as baryon spectrum and its $\bar{q}q$ content [4]. The constituent quark solitons of baryons were uncovered taking into account the both bosonized flavor and color degrees of freedom [6]. In particular, the study of meson-baryon scattering and resonances is a nontrivial task for unequal quark masses even in 2D [7].

Recently, in QCD$_4$ there appeared some puzzles related with unequal quark masses [8] providing an extra motivation to consider QCD$_2$ as a testing ground for non-perturbative methods that might have relevance in the real world. Claims for the existence of exotic baryons - that can not be composed of just three quarks - have inspired intense studies of the theory and phenomenology of QCD in the strong-interaction regime. In particular, it has led to the discovery that the strong coupling regime may contain unexpected correlations among groups of two or three quarks and antiquarks. Results of growing number of experiments at laboratories around the world provide contradictive situation regarding the experimental observation of possible pentaquark states, see e.g. [9]. These experiments have thus opened new lines of theoretical investigation that may survive even if the original inspiration - the exotic $\Theta^+$ pentaquark existence- is not confirmed. After the reports of null results started to accumulate the initial optimism declined, and the experimental situation remains ambiguous to the present. The increase in statistics led to some recent new claims for positive evidence [10], while the null result [11] by CLAS is specially significant because it contradicts their earlier positive result, suggesting that at least in their case the original claim was an artifact due to low statistics. All this experimental activity spurred a great amount of theoretical work in all kinds of models for hadrons and a renewed interest in soliton models. Recently, there is new strong evidence of an extremely narrow $\Theta^+$ resonance from DIANA collaboration and a very significant new evidence from LEPS. According to Diakonov, "the null results from the new round of CLAS experiments are compatible with what one should expect based on the estimates of production cross sections" [12].

It has been conjectured that the low-energy action of QCD$_2$ ($e_c \gg M_q$, $M_q$ quark mass and $e_c$ gauge coupling) might be related to massive two dimensional integrable models, thus leading to the exact solution of the strong coupled QCD$_2$ [4]. As an example of this picture, it has been shown that the so-called $su(2)$ affine Toda model coupled to matter (Dirac) field (ATM) [13] describes the low-energy spectrum of QCD$_2$ (one flavor and $N_C$ colors) [14]. The ATM model allowed the exact computation of the string tension in QCD$_2$ [14], improving the approximate result of [15]. The strong coupling sector of the $su(2)$ ATM model is described by the usual sine-Gordon model [16 – 18]. The baryons in QCD may be described as solitons in the bosonized formulation. In the strong-coupling limit the static classical soliton which describes a baryon in QCD$_2$ turns out to be the ordinary sine-Gordon kink, i.e.

$$\Phi(x) = \frac{4}{\beta_0} \tan^{-1}[\exp \beta_0 \sqrt{2}\widetilde{m}x] \tag{1.1}$$

where $\beta_0 = \sqrt{\frac{4\pi}{N_C}}$ is the coupling constant of the sine-Gordon theory, $8\sqrt{2}\widetilde{m}/\beta_0$ is the mass

of the soliton, and $\widetilde{m}$ is related to the common bare mass of the quarks by a renormalization group relation relevant to two dimensions. The soliton in (1.1) has non-zero baryon number as well as Y charge. The quantum correction to the soliton mass, obtained by time-dependent rotation in flavor space, is suppressed by a factor of $N_C$ compared to the classical contribution to the baryon mass [4]. The considerations of more complicated mass matrices and higher order corrections to the $M_q/e_c \to 0$ limit are among the issues that deserve further attention.

In this context, we show that various aspects of the low-energy effective QCD$_2$ action with unequal quark masses can be described by the (generalized) sine-Gordon model (GSG). The GSG model has appeared in the study of the strong coupling sector of the $sl(n, \mathbb{C})$ ATM theory [19 – 21], and in the bosonized multiflavor massive Thirring model [22]. In particular, the GSG model provides the framework to obtain (multi-)soliton solutions for unequal quark mass parameters. Choosing the normalization such that quarks have baryon number $Q_B^0 = 1$ and a one-soliton has baryon number $N_C$, we classify the configurations in the GSG model with baryon numbers $N_C, 2N_C, \ldots 4N_C$. For example, the double sine-Gordon model provides a kink type solution describing a multi-baryon state with baryon number $4N_C$ (see appendix). Then, using the GSG model we generalize the results of refs. [4, 5] which applied the semi-classical quantization method in order to uncover the normal [4] and exotic baryon [5] spectrum of QCD$_2$. One of the main features of the GSG model is that the one-soliton solution requires the QCD quark mass parameters to satisfy certain relationship. In two dimensions there are no spin degrees of freedom, so, the lowest-lying baryons are related to the purely symmetric Young tableau, the **10** dimensional representation of flavor SU(3). This is the analogue of the multiplet containing the baryons $\Delta$, $\Sigma$, $\Xi$, $\Omega^-$ in QCD$_4$. The next state corresponds to a state with the quantum numbers of four quarks and an antiquark, the so-called pentaquark, which in two dimensions forms a **35** representation of flavor SU(3). This corresponds to the four dimensional multiplet $\overline{\mathbf{10}}$, which contain the exotic baryons $\Theta^+$, $\bar{\Sigma}$, $\Xi^{--}$.

Here we improve the results of refs. [4, 5], such as the normal and exotic baryon masses, the relevant mass ratios and the radius parameter of the exotic baryons. The semi-classical computations of the masses get quantum corrections due to the unequal mass term contributions and to the form of the diagonal ansatz taken for the flavor field (related to GSG model) describing the lowest-energy state of the effective action. The corrections to the normal baryon masses are an increase of 3.5% to the earlier value obtained in [5], and in the case of the exotic baryon our computations improve the behavior of the quantum correction by decreasing the earlier value in 0.34 units, so making the semi-classical result more reliable. Let us mention that for the first exotic baryon [5] the quantum correction was greater than the classical term by a factor of 2.46, so that semi-classical approximation may not be a good approximation. As a curiosity, with the relevant values obtained by us for QCD$_2$ we computed the ratio between the lowest exotic baryon and the $R = \mathbf{10}$ baryon masses $M_{\mathbf{35}}/M_{\mathbf{10}} \sim 1.65$, which is only 1% larger than the analogous four dimensional QCD ratio $M_{\Theta+}/M_{\text{nucleon}} \sim 1.63$. In [5] the relevant QCD$_2$ ratio was 17% larger than this value. The mass formulae for the normal and exotic baryons corresponding, respectively, to the representations **10** and **35**, in two dimensions resemble the general chiral-soliton model

formula in four dimensions [23] except that there is no spin-dependent term $\sim J(J+1)$, and an analog term containing the soliton moment of inertia emerges.

The paper is organized as follows. The next section summarizes the bosonized low-energy effective action of QCD$_2$ and introduces the lowest-energy state described by the GSG action. The global QCD$_2$ symmetries are discussed. In section 3 the semi-classical method of quantization relevant to a general diagonal ansatz is introduced. In subsection 3.1 we briefly review the ordinary sine-Gordon soliton semi-classical quantization in the context of QCD$_2$. In section 4 we discuss the quantum correction to the SU(3) GSG ansatz in the framework of semi-classical quantization. In subsection 4.1 the GSG one-soliton state is rotated in SU(3) flavor space by a time- dependent $A(t)$. In subsection 4.2 the lowest-energy baryon state with baryon number $N_C$ is introduced. The possible vibrational modes are briefly discussed in subsection 4.3. Section 5 discusses the first and higher multiplet exotic baryons and provides the relevant quantum corrections the their masses, the ratio $M_{\bf 35}/M_{\bf 10}$, and an estimate for the exotic baryon radius parameter. The last section presents a summary and some discussions. In the appendix we provide the GSG solitons and kink solutions relevant to our QCD$_2$ discussion.

## 2. Baryons in bosonized QCD$_2$

### 2.1 The bosonized effective action

The QCD$_2$ action is written in terms of gauge fields $A_\mu$ and fundamental quark fields $\psi$ as

$$S_F[\psi, A_\mu] = \int d^2x \left\{ -\frac{1}{2e_c^2} Tr(F_{\mu\nu} F^{\mu\nu}) - \bar{\psi}^{ai}[(i\partial\!\!\!/ + A\!\!\!/)]\psi_{ai} + \mathcal{M}_{ij}\bar{\psi}^{ai}\psi_{aj} \right\}, \tag{2.1}$$

where $a$ is the color index ($a = 1, 2, \ldots, N_C$) and $i$ the flavor index ($i = 1, 2, \ldots, N_f$), $e_c$, with dimension of a mass, is the quark coupling to the gauge fields, the matrix $\mathcal{M}_{ij} = m_i \delta_{ij}$ ($m_i$ being the quark masses) takes into account the quark mass splitting, and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the gauge field strength.

The bosonized action in the strong-coupling limit ($e_c \gg$ all $m_i$) becomes [4, 6]

$$S_{\text{eff}}[g] = N_c S[g] + m^2 N_m \int d^2x Tr_f \left[ \mathcal{D}(g + g^\dagger) \right], \tag{2.2}$$

where $g$ is a matrix representing $U(N_f)$, $\mathcal{D} = \frac{\mathcal{M}}{m_0}$, $m_0$ is an arbitrary mass parameter and the effective mass scale $m$ is given by

$$m = \left[ N_c c m_0 \left( \frac{e_c \sqrt{N_F}}{\sqrt{2\pi}} \right)^{\Delta_c} \right]^{\frac{1}{1+\Delta_c}}, \tag{2.3}$$

with

$$\Delta_c = \frac{N_c^2 - 1}{N_c(N_c + N_F)}. \tag{2.4}$$

In (2.2) $S[g]$ is the WZNW action and $N_m$ stands for normal ordering with respect to $m$. In the large $N_c$ limit, which we use below to justify the semi-classical approximation, the scale $m$ becomes

$$m = 0.59 N_F^{\frac{1}{4}} \sqrt{N_c e_c m_0}, \tag{2.5}$$

so, $m$ takes the value $0.77\sqrt{N_c e_c m_0}$ for three flavors. Notice that we first take the strong-coupling limit $e_c \gg \text{all} \quad m_i$, and then take $N_c$ to be large, thus it is different from the 't Hooft limit [3], where $e_c^2 N_c$ is held fixed.

Following the Skyrme model approach it is useful to first ask for classical soliton solutions of the bosonic action which are heavy in the $N_C \rightarrow$ large limit. The action (2.2) is a massive WZNW action and possesses the property that if $g$ is non-diagonal it can not be a classical solution, as after a diagonalization to

$$g_0 = \text{diag}(e^{-i\beta_0\Phi_1(x)}, \, e^{-i\beta_0\Phi_2(x)}, \, \dots \, e^{-i\beta_0\Phi_{N_f}(x)}), \quad \sum_i \Phi_i(x) = \phi(x), \quad \beta_0 \equiv \sqrt{\frac{4\pi}{N_C}} \quad (2.6)$$

it will have lower energy [24]. Thus, the minimal energy solutions of the massive WZNW model are necessarily in a diagonal form. The majority of particles given by (2.6) are not going to be stable, but must decay into others.

Previous works consider the diagonal form (2.6) such that the action (2.2) reduces to a sum of $N_f$ *independent* ordinary sine-Gordon models, each one for the corresponding $\Phi_i$ field and parameters

$$\widetilde{m}_i^2 = \frac{m_i}{m_0} m^2. \quad (2.7)$$

In this approach the lowest lying baryon is represented by the minimum-energy configuration for this class of ansatz, i.e.

$$\hat{g}_0(x) = \text{diag}\Big(1, \, 1, \, \dots, \, e^{-i\sqrt{\frac{4\pi}{N_C}}\Phi_{N_f}}\Big), \quad (2.8)$$

with $m_{N_f}$ chosen to be the smallest mass.

In this paper we will consider the ansatz (2.6) for

$$N_f = \frac{n}{2}(n-1), \quad N_f \equiv \text{number of positive roots of } su(n), \quad (2.9)$$

such that $\frac{(n-2)(n-1)}{2}$ linear constraints are imposed on the fields $\Phi_i$. This model corresponds to the generalized sine-Gordon model (GSG) recently studied in the context of the bosonization of the so-called generalized massive Thirring model (GMT) with $N_f$ fermion species [19, 20, 22]. The classical GSG model and some of its properties, such as the algebraic construction based on the affine $sl(n, \mathbb{C})$ Kac-Moody algebra and the soliton spectrum has been the subject of a recent paper [21].

The WZ term in (2.2) vanishes for either static or diagonal solution, so, for the ansatz (2.6) and after redefining the additive constant term the action becomes

$$S[g_0] = \int d^2x \sum_{i=1}^{N_f} \left[\frac{1}{2}(\partial_\mu \Phi_i)^2 + 2\widetilde{m}_i^2\Big(\cos\beta_0\Phi_i - 1\Big)\right], \quad (2.10)$$

with coupling $\beta_0$ and mass parameters $\widetilde{m}_i$ defined in (2.6) and (2.7), respectively.

The $\Phi_i$ fields in (2.10) satisfy certain constraints of the type

$$\Phi_p = \sum_{i=1}^{n-1} \sigma_{pi}\Phi_i, \quad p = n, n+1, \dots, N_f \quad (2.11)$$

where $\sigma_{pi}$ are some constant parameters. From the Lie algebraic construction of the GSG model these parameters arise from the relationship between the positive and simple roots of $su(n)$. Even though our treatment until section 3 is valid for any $N_f$, starting in section 4 we will concentrate on the $N_f = 3$ case.

It is interesting to recognize the similarity between the potential of the model (2.10)–(2.11) for the $N_f = 3$ case [in su(3) GSG model one has $n = N_f = 3$ and just one constraint equation in (2.11)] and the effective chiral Lagrangian proposed by Witten to describe low-energy behavior of four dimensional QCD [25]. In Witten's approach the potential term reads

$$V^{\text{Witten}}(U) = f_\pi^2 \left[ -\frac{1}{2} Tr M(U + U^\dagger) + \frac{k}{2N_C}(-i\ln\text{Det}\,U - \theta)^2 \right], \qquad (2.12)$$

where $U$ is the pseudoscalar field matrix and $M = \text{diag}\Big(m_u; m_d; m_s\Big)$ is the quark mass matrix. Phenomenologically $m_{\eta'}^2 \gg m_\pi^2$, $m_K^2$, $m_\eta^2$, implying that $\frac{k}{N_C} > b\,m_s \gg b\,m_u$, $b\,m_d$ [the parameter $b$ is $O(\Lambda)$, where $\Lambda$ is a hadronic scale). Because $M$ is diagonal, one can look for a minimum of $V^{\text{Witten}}(U)$ in the form $U = \text{diag}\Big(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}\Big)$. Since the second term dominates over the first, one has $\sum \phi_j = \theta$ up to the first approximation. So, choosing $\theta = 0$, (2.12) reduces to a model of type (2.10)–(2.11) defined for $N_f = 3$. This is the $sl(3)$ GSG model, which possesses soliton and kink type solutions (see the appendix), and will be the main ingredient of our developments in sections 4 and 5.

The potential term in (2.10) is invariant under

$$\Phi_i \rightarrow \Phi_i + \frac{1}{\beta_0} 2\pi N_i, \ (N_i \in \mathbb{Z}). \qquad (2.13)$$

All finite energy configurations, whether static or time-dependent, can be divided into an infinite number of topological sectors, each characterized by a set

$$\Big[n_1, n_2, \ldots, n_{N_f}\Big] = \Big[(N_1^+ - N_1^-), (N_2^+ - N_2^-), \ldots, (N_{N_f}^+ - N_{N_f}^-)\Big] \qquad (2.14)$$

$$\Phi_i(\pm\infty) = \frac{1}{\beta_0} 2\pi N_i^\pm \qquad (2.15)$$

corresponding to the asymptotic values of the fields at $x = \pm\infty$. The $n_i's$ satisfy certain relationship arising from the constraints (2.11) and the invariance (2.13) (some examples are given in the appendix for the soliton and kink type solutions in the SU(3) case).

Conserved charges, corresponding to the vector current $J_{ij}^\mu = \bar{\psi}_i^a \gamma^\mu \psi_j^a$, can be computed as

$$Q^A[g(x)] = \int dx [J_0(\frac{T^A}{2})], \qquad (2.16)$$

where $(\frac{T^A}{2})$ are the $su(n)$ generators and the U(1) baryon number is obtained using the identity matrix instead of $(\frac{T^A}{2})$. For $g_0$ given in eq. (2.6) the baryon number of any given flavor $j$ is given by $\mathcal{Q}_B^j = N_c n_j$, so, the total baryon number becomes

$$\mathcal{Q}_B = N_C(n_1 + n_2 + \cdots + n_{N_f}), \qquad (2.17)$$

and the "hypercharge" is given by

$$
\begin{aligned}
Q_Y &= \frac{1}{2}\text{Tr} \int dx \Big( J_0 \lambda_{N_f^2-1} \Big) \\
&= \frac{1}{2} N_C \Big( n_1 + n_2 + \cdots + n_{N_f-1} - (N_f-1)n_{N_f} \Big) \sqrt{\frac{2}{N_f^2 - N_f}}.
\end{aligned}
\tag{2.18}
$$

The total baryon number is clearly an integer multiple of $N_C$. In the case of (2.8) they reduce to $\mathcal{Q}_B = N_C$ and $Q_Y^0 = -\frac{1}{2}\sqrt{2(N_f-1)/N_F}N_C$, respectively [for $\sqrt{4\pi/N_C}\Phi_{N_f}(+\infty) = 2\pi$, $\Phi_{N_f}(-\infty) = 0$] [4]. We are choosing the convention in which the quarks have baryon number $Q_B = 1$, so the soliton representing a physical baryon has baryon number $N_C$.

A global $U_V(N_f)$ transformation $\widetilde{g}_0 = Ag_0(x)A^{-1}$ is expected to turn on the other charges. Let us introduce

$$
A = \begin{pmatrix} z_1^{(1)} & \cdots & z_1^{(N_f)} \\ z_2^{(1)} & \cdots & z_2^{(N_f)} \\ z_{N_f}^{(1)} & \cdots & z_{N_f}^{(N_f)} \end{pmatrix},
\tag{2.19}
$$

$$
\sum_{p=1}^{N_f} z_p^{(i)} z_p^{(j)\star} = \delta_{ij}.
\tag{2.20}
$$

Now

$$
\widetilde{g}_0 = \sum_{j=1}^{N_f} e^{i\beta_j \Phi_j} \mathbb{Z}^{(j)}, \qquad \mathbb{Z}_{pq}^{(j)} = z_p^{(j)} z_q^{(j)\star},
\tag{2.21}
$$

The charges with $\widetilde{g}_0$ are

$$
(\widetilde{Q}^0)^A = \frac{1}{2} N_C \text{Tr} \sum_i \Big( n_i T^A \mathbb{Z}^{(i)} \Big)
\tag{2.22}
$$

The baryon number is unchanged. The $U(n)$ possible representations will be discussed below in the semi-classical quantization approach.

# 3. Semi-classical quantization and the GSG ansatz

In order to implement the semi-classical quantization let us consider

$$
g(x,t) = A(t)g_0(x)A^{-1}(t), \quad A(t) \in U(N_f)
\tag{3.1}
$$

and derive the effective action for $A(t)$ by substituting $g(x,t)$ into the original action. So, following similar steps to the ones developed in [4] one can get

$$
\widetilde{S}(g(x,t)) - \widetilde{S}(g_0(x)) = \frac{N_C}{8\pi} \int d^2x \text{Tr}\Big( \Big[ A^{-1}\dot{A}, g_0 \Big] \Big[ A^{-1}\dot{A}, g_0^\dagger \Big] \Big) +
$$
$$
\frac{N_C}{2\pi} \int d^2x \text{Tr}\{ (A^{-1}\dot{A})(g_0^\dagger \partial_x g_0) \} + m^2 \text{Tr} \int d^2x [(\mathcal{D}Ag_0A^\dagger - \mathcal{D}g_0) + c.c.]
\tag{3.2}
$$

The action above for $\mathcal{D}_{ij} = \delta_{ij}$ (in this case the last integrand after taking the trace operation vanishes identically) is invariant under global $\mathrm{U}(N_f)$ transformation

$$A \to UA, \tag{3.3}$$

where $U \in G = \mathrm{U}(N_f)$. This corresponds to the invariance of the original action (with mass of the same magnitude for all flavors) under $g \to UgU^{-1}$. It is also invariant under the local changes

$$A(t) \to A(t)V(t), \tag{3.4}$$

where $V(t) \in H$. This subgroup $H$ of $G$ is nothing but the invariance group of $g_0$. Below we will find some particular cases of $H$.

We define the Lie algebra valued variables $q^i$, $y_a$ through $A^{-1}\dot{A} = i\sum\{\dot{q}_i E_{\alpha_i} + \dot{y}_a H^a\}$ in the generalized Gell-Mann representation [26]. In terms of these variables the action (3.2), for a diagonal mass matrix such that $\mathcal{D}_{ij} = \delta_{ij}$, takes the form

$$S[q,y] = \int dt \{\sum_{i=1}^{N_f} \frac{1}{2M_i}\dot{q}_i\dot{q}_{-i} - \sum_{a=1}^{N_f-1}\sqrt{\frac{2}{(a+1)^2-(a+1)}} \times$$
$$\times (n_1 + n_2 + \cdots + n_a - an_{a+1})\dot{y}_a\}, \tag{3.5}$$

where $q_{\pm i}$ are associated to the positive and negative roots, respectively, and

$$\frac{1}{2M_i} = \frac{N_C}{2\pi}\int_{-\infty}^{+\infty}[1 - \cos\beta_0\Phi_i], \quad \Phi_i \neq 0. \tag{3.6}$$

In the case of vanishing $\Phi_j \equiv 0$ for a given $j$ one must formally set $M_j = +\infty$ in the relevant terms throughout.

In the case of $\hat{g}_0 = \mathrm{diag}(1,1,\ldots,e^{i\beta_{N_f}\Phi_{N_f}})$ the second summation in (3.5) reduces to the unique term $[-N_c\sqrt{\frac{2(N_f-1)}{N_f}}\dot{y}_{N_f-1}]$ [4].

When written in terms of the general diagonal field $g_0(x)$ and the $\mathrm{U}(N_f)$ field $A(t)$, the charges associated to the global $\mathrm{U}(N_f)$ symmetry, (2.16), become

$$Q^B = i\frac{N_C}{8\pi}\int dx \mathrm{Tr}\{T^B A\{(g_0^\dagger\partial_x g_0 - g_0\partial_x g_0^\dagger) + [g_0, [A^{-1}\dot{A}, g_0^\dagger]]\}A^{-1}\} \tag{3.7}$$

A convenient parameterization, instead of the parameters used in (3.5), is (2.19) since in the above expressions, for $Q^B$ and the action (3.2), there appear the fields $A, A^{-1}$, as well as their time derivatives. Now, for a diagonal mass matrix such that $\mathcal{D} = \frac{m_i}{m_0}\delta_{ij}$, the expression (3.2) can be written in terms of the variables $z_p^{(i)}$ subject to the relationships (2.20)

$$\widetilde{S}(g(x,t)) - \widetilde{S}(g_0(x)) = S[z_p^{(i)}(t), \Phi_i(x)] \tag{3.8}$$

$$S[z_p^{(i)}(t), \Phi_i(x)] = \frac{N_C}{2\pi}\int d^2x \sum_{p,q;\,i<j}[\cos(\beta_i\Phi_i - \beta_j\Phi_j) - 1][\dot{z}_p^{(i)}z_q^{(i)\star}\dot{z}_q^{(j)}z_p^{(j)\star}] -$$
$$i\frac{N_C}{2\pi}\int d^2x \sum_{i,p}\beta_i\partial_x\Phi_i\dot{z}_p^{(i)}z_p^{(i)\star} +$$
$$\int dt\, 2[\sum_{i,p}\cos(\beta_i\Phi_i)\widetilde{m}_p^2 z_p^{(i)}z_p^{(i)\star} - \sum_i\cos(\beta_i\Phi_i)\widetilde{m}_i^2] \tag{3.9}$$

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Let us choose the index $k$ corresponding to the smallest mass $m_k$. So, integrating over $x$ in (3.9) we may write

$$
\begin{aligned}
S[z_p^{(i)}(t)] = -\frac{1}{2}\int dt \sum_{i<j}^{N_f}\sum_{p,q}^{N_f} M_{ij}^{-1}\dot{z}_p^{(i)} z_q^{(i)\,\star}\dot{z}_q^{(j)} z_p^{(j)\,\star} - \\
i\frac{N_C}{2}\int dt \sum_i n_i\Big[\dot{z}_p^{(i)} z_p^{(i)\,\star} - z_p^{(i)}\dot{z}_p^{(i)\,\star}\Big] - \\
\frac{2\pi}{N_c}\int dt\Big\{ \sum_{i,\,p}\Big[\frac{\widetilde{m}_p^2}{M_i} - \frac{\widetilde{m}_k^2}{M_k}\Big] z_p^{(i)} z_p^{(i)\,\star} + \frac{2\pi}{N_c}\Big[\sum_i \frac{\widetilde{m}_i^2}{M_i} - \frac{\widetilde{m}_k^2}{M_k}N_\phi\Big]\Big\} \\
+ \int dt (z_p^{(i)} z_p^{(j)\,\star} - \delta_{ij})\lambda_{ij}
\end{aligned}
\tag{3.10}
$$

where $N_\phi$ is the number of nonvanishing $\Phi_i$ fields and we have introduced some Lagrange multipliers enforcing the relationships (2.20). The constants $M_{ij}$ above are defined by

$$
\frac{1}{2M_{ij}} \equiv \frac{N_C}{2\pi}\int dx[1 - \cos(\beta_0\Phi_i - \beta_0\Phi_j)]; \quad i < j.
\tag{3.11}
$$

If the field solutions are such that $\Phi_i = \Phi_j$, then one must set formally $M_{ij} \to +\infty$ in place of the corresponding constants.

Likewise, we can write the $\mathrm{U}(N_f)$ charges, eq. (3.7), in terms of the $z_p^{(i)}$ variables

$$
\begin{aligned}
Q^A &= \frac{1}{2}T_{\beta\alpha}^A Q_{\alpha\beta}, \\
Q_{\alpha\beta} &= N_C \sum_j n_j\, z_\alpha^{(j)} z_\beta^{(j)\,\star} - \frac{i}{2}\sum_{i,j} M_{ij}^{-1} z_\alpha^{(j)} z_\gamma^{(j)\,\star}\dot{z}_\gamma^{(i)} z_\beta^{(i)\,\star}.
\end{aligned}
\tag{3.12}
$$

The second $\mathrm{U}(N_f)$ Casimir operator is obtained from the charge matrix elements $Q_{\alpha\beta}$

$$
\begin{aligned}
Q^A Q_A &= \frac{1}{2}Q_{\alpha\beta}Q_{\beta\alpha}, \\
&= \frac{1}{2}N_C^2\sum_i n_i n_i - \frac{1}{4}\sum_{i<j}\left(M_{ij}^{-1}\right)^2\dot{z}_\alpha^{(j)} z_\beta^{(j)\,\star}\dot{z}_\beta^{(i)} z_\alpha^{(i)\,\star}
\end{aligned}
\tag{3.13}
$$

The expressions above greatly simplify in certain particular cases of the ansatz (2.6), the ansatz (2.8) has been studied extensively in the literature before. In the next subsection we review this case and in further sections we analyze the semiclassical quantization of the GSG ansatz given for $N_f = 3$ flavors.

## 3.1 Review of usual sine-Gordon soliton and baryons in QCD$_2$

In this subsection we briefly review the formalism applied to the ansatz (2.8), which is related to the usual SG one-soliton as the lowest baryon state. In order to calculate the quantum correction it is allowed the sine-Gordon soliton to rotate in $\mathrm{SU}(N_f)$ space by a time dependent matrix $A(t)$ as in (3.1). Let us consider the single baryon state defined for

the ansatz (2.8) for the sine-Gordon soliton solution $\Phi_{N_f} \equiv \Phi_{1-soliton}$ [ $\Phi_{1-soliton}$ is given by (1.1)]; so, in the relations above one must set

$$n_{N_f} = 1; \quad n_j = 0 \,(j \neq N_f); \quad M_{jk}^{-1} \equiv 0 \,(j < k < N_f); \quad M_{jN_f}^{-1} \equiv M_{N_f}^{-1} \,(j < N_f), \quad (3.14)$$

where $M_{N_f}^{-1}$ can be computed using eq. (3.6) for $i = N_f$ for the soliton (1.1)

$$\frac{1}{2M_{N_f}} = \frac{1}{\sqrt{2}\,\widetilde{m}}\Big(\frac{N_C}{\pi}\Big)^{3/2} \quad (3.15)$$

Then, for the ansatz (2.8), i.e. $\hat{g}_0(x) = \mathrm{diag}\Big(1,\,1,\,\ldots,\,e^{-i\sqrt{\frac{4\pi}{N_C}}\Phi_{N_f}}\Big)$, the effective action (3.10) can be written as

$$\begin{aligned}
S[z_j^{(N_f)}(t)] = {} & \frac{1}{2M_{N_f}} \int dt [\dot{z}_j^{(N_f)\star}\dot{z}_j^{(N_f)} - (z_i^{(N_f)\star}\dot{z}_i^{(N_f)})(\dot{z}_k^{(N_f)\star}z_k^{(N_f)})] \\
& - \frac{2\pi}{M_{N_f}N_C} \int dt \sum_{i=1}^{N_f} \Big(\widetilde{m}_i^2 - \widetilde{m}_{N_f}^2\Big)z_i^{(N_f)\star}z_i^{(N_f)} \\
& -i\frac{N_C}{2} \int dt\, n_{N_f} \Big(z_j^{(N_f)\star}\dot{z}_j^{(N_f)} - \dot{z}_j^{(N_f)\star}z_j^{(N_f)}\Big) \\
& + \int dt [(z_p^{(N_f)}z_q^{(N_f)\star} - \delta_{pq})\lambda^{pq}], \quad (3.16)
\end{aligned}$$

where $n_{N_f} = 1$, $M_{N_f}$ is given by (3.15) and $m_{N_f}$ entering $\widetilde{m}_{N_f}$ is chosen to be the smallest quark mass. Notice that for equal quark masses the second line in eq. (3.16) vanishes identically. According to (3.3)–(3.4), the symmetries of $S[z_j^{(N_f)}(t)]$ are the global $\mathrm{U}(N_f)$ group (for equal quark masses) under which

$$z_\alpha^{(N_f)} \to z_\alpha^{'(N_f)} = U_{\alpha\beta}z_\beta^{(N_f)}, \quad U \in \mathrm{U}(N_f), \quad (3.17)$$

and a local $\mathrm{U}(1)$ subgroup of $H$ under which

$$z_\alpha^{(N_f)} \to z_\alpha^{'(N_f)} = e^{i\delta(t)}z_\alpha^{(N_f)}. \quad (3.18)$$

The action (3.16) has been considered in order to find the quantum correction to the soliton mass for certain representations $R$ of the flavor symmetry $\mathrm{SU}(N_f)$. The case of equal quark masses has been studied in the literature [4, 5, 7, 27]. Certain properties in the case of different quark masses have been considered in [6] for the ansatz (2.8).

In this approach the minimum-energy configuration for the class of ansatz (2.8), with $m_{N_f}$ the smallest mass, corresponds to the state of lowest-lying baryon [4] which in the large-$N_C$ limit possesses the classical mass

$$M_{\mathrm{baryon}}^{cl} = 4\widetilde{m}_{N_f}\Big(\frac{2N_C}{\pi}\Big)^{1/2} \approx 1.90N_f^{1/4}\sqrt{e_c m_{N_f}}N_C, \quad (3.19)$$

where $\widetilde{m}_{N_f}$ has been given in (2.7) for $i = N_f$.

Moreover, for the Ansatz (2.8) the $\mathrm{SU}(N_f)$ charges become

$$Q_{\alpha\beta} = N_C n_{N_f} z_\alpha^{(N_f)} z_\beta^{(N_f)\star} + \frac{i}{2M_{N_f}} \left[ z_\alpha^{(N_f)} z_\beta^{(N_f)\star} \left( \dot{z}_\delta^{(N_f)} z_\delta^{(N_f)\star} - z_\delta^{(N_f)} \dot{z}_\delta^{(N_f)\star} \right) \right.$$
$$\left. + z_\alpha^{(N_f)} \dot{z}_\beta^{(N_f)\star} - \dot{z}_\alpha^{(N_f)} z_\beta^{(N_f)\star} \right] \tag{3.20}$$

The corresponding second Casimir can be obtained from (3.13)

$$Q_A Q^A = \frac{1}{2} Q_{\alpha\beta} Q_{\beta\alpha} = \frac{1}{2} N_C^2 n_{N_f}^2 + \frac{1}{4M_{N_f}^2} \left( Dz \right)_\alpha^\dagger \left( Dz \right)_\alpha, \quad Dz \equiv \dot{z} - z(z^\dagger \dot{z}) \tag{3.21}$$

Moreover, denoting the $\mathrm{SU}(N_f)$ second Casimir operator by $C_2(N_f)$ one can write

$$Q_A Q^A = C_2(N_f) + \frac{1}{2N_f} (\mathcal{Q}_B)^2, \tag{3.22}$$

where $\mathcal{Q}_B$ is the baryon number (2.17), which in this case reduces to $\mathcal{Q}_B = N_C$.

In the case of a single baryon given by $\hat{g}_0$, eq. (2.8), and for unequal quark masses, the hamiltonian is linear in the quadratic Casimir operator. To see this we now derive the hamiltonian corresponding to the action (3.16). The canonical momenta are given by

$$p_\alpha = \frac{\partial L}{\partial \dot{z}_\alpha^{(N_f)\star}} = \frac{1}{2M_{N_f}} \left[ \dot{z}_\alpha^{(N_f)} - \left( \dot{z}_\beta^{(N_f)} z_\beta^{(N_f)\star} \right) z_\alpha^{(N_f)} \right] + \frac{iN_C}{2} z_\alpha^{N_f} \tag{3.23}$$

and there is a conjugate expression for $p_\alpha$. Therefore, from $H = p_\alpha \dot{z}_\beta^{(N_f)\star} + p_\alpha^\star \dot{z}_\beta^{(N_f)} - L$, one can get the hamiltonian

$$H = \frac{1}{2M_{N_f}} \left( Dz \right)_\alpha^\dagger \left( Dz \right)_\alpha + \frac{2\pi}{M_{N_f} N_C} \sum_{i=1}^{N_f} \left( \widetilde{m}_i^2 - \widetilde{m}_{N_f}^2 \right) z_i^{(N_f)\star} z_i^{(N_f)}. \tag{3.24}$$

However, one must take a proper care of the relevant constraint (2.20) which was incorporated through the addition of a Lagrange multiplier in the action (3.16). A proper treatment of a constrained system must be performed at this point [4]. In [4, 27] it was shown that the local $\mathrm{U}(1)$ gauge symmetry (3.18) leads to the constraint

$$Q_{N_f N_f} = 0 \Rightarrow \mathcal{Q}_B = \sqrt{2N_f(N_f - 1)} Q_Y \tag{3.25}$$

which has to be imposed on physical states. This implies that the representation $R$ must contain a state with $Y$ charge

$$\bar{Q}_Y = \sqrt{\frac{1}{2N_f(N_f - 1)}} N_C. \tag{3.26}$$

The remaining states will be generated through the application of the $\mathrm{SU}(N_f)$ transformations to this one. For states with only quarks and no antiquarks, the condition that $\mathcal{Q}_B = N_C$ implies that only representations described by Young tableaux with $N_C$ boxes appear. The additional constraint that $Q_Y = \bar{Q}_Y$ implies that all $N_C$ quarks belong to

SU($N_f - 1$), i.e., this state does not involve the $N_f'^{th}$ quark flavor. These constraints are automatically satisfied in the totally symmetric representation of $N_C$ boxes, which is the only representation possible in two dimensions. This is because the state wave functions have to be constructed out of the components of the complex vector $z^{(N_f)}$ as

$$\psi(z^{(N_f)}, z^{(N_f)\star}) = (z_1^{(N_f)})^{p_1} \ldots (z_{N_f}^{(N_f)})^{p_{N_f}} (z_1^{(N_f)\star})^{q_1} \ldots (z_{N_f}^{(N_f)\star})^{q_{N_f}} \qquad (3.27)$$

with $\sum_{i=1}^{N_f}(p_i - q_i) = N_C$.

The lowest such multiplet has

$$\sum_{i=1}^{N_f} p_i = N_C \quad \text{and all} \quad q_i = 0. \qquad (3.28)$$

This multiplet corresponds to the Young tableaux

$$\overbrace{\square \cdots \square}^{N_c} \qquad (3.29)$$

In QCD$_2$ for $N_C = 3$, $N_F = 3$ we get only the **10** of SU(3).

Then, taking into account (3.21), (3.22) and (2.7), the expression (3.24) becomes [4, 6]

$$H = M_{\text{baryon}}^{cl} \left\{ 1 + \left(\frac{\pi}{2N_C}\right)^2 \left[ C_2(R) - \frac{n_{N_f}^2 N_C^2}{2N_f}(N_f - 1) \right] + \sum_{i=1}^{N_f} \frac{m_i - m_{N_f}}{m_{N_f}} |z_i^{(N_f)}|^2 \right\}, \quad (3.30)$$

where $M_{\text{baryon}}^{cl}$ is given by (3.19) and $C_2(R)$ is the value of the quadratic Casimir for the flavor representation $R$ of the baryon. For a baryon state given by SG 1-soliton solution one must set $n_{N_f} = 1$ in the hamiltonian above. Notice that the Hamiltonian depends on $m_0$ only through $M_{\text{baryon}}^{cl}$, so the overall mass scale is undetermined, only the mass ratios are meaningful. The mass term contributions come from quantum fluctuations around the classical soliton, consistency with the semi-classical approximation requires that it be very small compared to one. However, these terms vanish for equal quark masses [4, 5]. The **10** baryon mass becomes

$$M(baryon) = M_{\text{classical}} \left[ 1 + \left(\frac{\pi^2}{8}\right) \frac{N_f - 1}{N_C} \right]. \qquad (3.31)$$

Notice that the quantum correction is suppressed by a factor of $N_C$. Moreover, the quantum correction for $N_C = 3$, $N_f = 3$ numerically becomes $\sim 0.82$.

The hamiltonian (3.30) taken for equal quark masses has been used to compute the energy of the first exotic baryon $\mathcal{E}_1$ (a state containing $N_C + 1$ quarks and just one anti-quark) by taking the corresponding Casimir $C_2(\mathcal{E}_1)$ for $R = \mathbf{35}$ of flavor relevant to the exotic state [5]. For further analysis we record the mass of this exotic baryon

$$M(\mathcal{E}_1) = M(classical) \left[ 1 + \frac{\pi^2}{8} \frac{1}{N_C} \left( 3 + N_f - \frac{6}{N_f} \right) + \frac{3\pi^2}{8} \frac{1}{N_C^2} \left( N_f - \frac{3}{N_f} \right) \right]. \quad (3.32)$$

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In the interesting case $N_C = 3$, $N_f = 3$ this becomes

$$M(\mathbf{35}) = M(classical)\left\{1 + \frac{\pi^2}{4}\right\}. \tag{3.33}$$

In this case the correction due to quantum fluctuations around the classical solution is larger than the classical term. So, the semi-classical approximation may not be a good approximation. However, observe that the ratio $M(\mathbf{35})/M(\mathbf{10}) \sim 1.9$, which is 17% larger than the ratio between the experimental masses of the $\Theta^+$ and the nucleon. See more on this point below. These semi-classical approximations may be improved by introducing different ansatz for $g_0$ and considering unequal quark mass parameters. These points will be tackled in the next sections.

## 4. The GSG model, the unequal quark masses and baryon states

In the following we will concentrate on the effective action (3.10) for the particular case $N_f = 3$ and unequal quark mass parameters. So, the SU(3) flavor symmetry is broken explicitly by the mass terms.

The effective Lagrangian in the case of $N_f = 3$ from (3.10), upon using (2.20), can be written as

$$\begin{aligned}
S[z_p^{(i)}(t)] = \frac{1}{4} \int dt &\left\{ \left(M_{12}^{-1} + M_{13}^{-1} - M_{23}^{-1}\right)\left[\dot{z}_\alpha^{(1)} \dot{z}_\alpha^{(1)\,\star} - \dot{z}_\alpha^{(1)} z_\alpha^{(1)\,\star} z_\beta^{(1)\,\star} \dot{z}_\beta^{(1)\,\star}\right] + \right. \\
&\left. \left(M_{12}^{-1} - M_{13}^{-1} + M_{23}^{-1}\right)\left[\dot{z}_\alpha^{(2)} \dot{z}_\alpha^{(2)\,\star} - \dot{z}_\alpha^{(2)} z_\alpha^{(2)\,\star} z_\beta^{(2)} \dot{z}_\beta^{(2)\,\star}\right] + \right. \\
&\left. \left(-M_{12}^{-1} + M_{13}^{-1} + M_{23}^{-1}\right)\left[\dot{z}_\alpha^{(3)} \dot{z}_\alpha^{(3)\,\star} - \dot{z}_\alpha^{(3)} z_\alpha^{(3)\,\star} z_\beta^{(3)} \dot{z}_\beta^{(3)\,\star}\right]\right\} - \\
& i\frac{N_C}{2} \int dt \sum_{i,p} n_i\left[\dot{z}_p^{(i)} z_p^{(i)\,\star} - z_p^{(i)} \dot{z}_p^{(i)\,\star}\right] - \\
& \int dt \left\{\frac{2\pi}{N_c} \sum_{i,p} \left[\frac{\widetilde{m}_p^2}{M_i} - \frac{\widetilde{m}_k^2}{M_k}\right] z_p^{(i)} z_p^{(i)\,\star} + \frac{2\pi}{N_c}\left[\sum_i \frac{\widetilde{m}_i^2}{M_i} - \frac{\widetilde{m}_k^2}{M_k} N_\phi\right]\right\} \tag{4.1}
\end{aligned}$$

From (3.13) and following similar steps the second U(3) Casimir operator can be written as

$$\begin{aligned}
Q^A Q_A &= \frac{1}{2} Q_{\alpha\beta} Q_{\beta\alpha}, \\
&= \frac{1}{2} N_C^2 \sum_j n_j n_j + \frac{1}{8}\left\{ \left(M_{12}^{-2} + M_{13}^{-2} - M_{23}^{-2}\right)\left[\dot{z}_\alpha^{(1)} \dot{z}_\alpha^{(1)\,\star} - \dot{z}_\alpha^{(1)} z_\alpha^{(1)\,\star} z_\beta^{(1)} \dot{z}_\beta^{(1)\,\star}\right] + \right. \\
&\left. \left(M_{12}^{-2} - M_{13}^{-2} + M_{23}^{-2}\right)\left[\dot{z}_\alpha^{(2)} \dot{z}_\alpha^{(2)\,\star} - \dot{z}_\alpha^{(2)} z_\alpha^{(2)\,\star} z_\beta^{(2)} \dot{z}_\beta^{(2)\,\star}\right] + \right. \\
&\left. \left(-M_{12}^{-2} + M_{13}^{-2} + M_{23}^{-2}\right)\left[\dot{z}_\alpha^{(3)} \dot{z}_\alpha^{(3)\,\star} - \dot{z}_\alpha^{(3)} z_\alpha^{(3)\,\star} z_\beta^{(3)} \dot{z}_\beta^{(3)\,\star}\right]\right\}. \tag{4.2}
\end{aligned}$$

As a particular case for the ansatz (2.8) let us take $N_f = 3$, so $n_1 = n_2 = 0$ in (2.14). In (3.11) one can set formally $M_{12} \equiv +\infty$ and in view of (3.6) the remaining parameters can be written as $M_{13} = M_{23} \equiv M_3$. Thus, taking into account these parameters the

expressions for the action (4.1) and the second Casimir (4.2) reduce to the well known ones (3.16) and (3.21), respectively.

Next, we discuss the action (4.1) and the second Casimir (4.2) operator for the soliton and kink type solutions of the GSG model. In appendices A.1, A.2 and A.3 we classify these type of solutions. There are three 1−soliton solutions [see eqs. (A.13), (A.20) and (A.28)] which correspond to baryon number $N_C$ [see eqs. (A.19), (A.26) and (A.33)], because the GSG model possesses the symmetry (A.11) the third soliton is doubly degenerated. From the fields relationships (A.17), (A.24) and (A.31) one has the three 1−soliton cases

$$i) \quad \Phi_1 = -\Phi_2 = \Phi_3 = \varphi_1 \Rightarrow M_{13} = +\infty, \; M_{12} = M_{23} \equiv \mathcal{M}_2; \; M_1 = M_2 = M_3 = \widetilde{M_2},$$
(4.3)

$$ii) \quad -\Phi_1 = \Phi_2 = \Phi_3 = \varphi_2 \;\; \Rightarrow \; M_{23} = +\infty, \; M_{12} = M_{13} \equiv \mathcal{M}_1; \; M_1 = M_2 = M_3 = \widetilde{M_1},$$
(4.4)

$$iii) \quad \Phi_1 = \Phi_2 = -\Phi_3 = \hat{\varphi} \;\; \Rightarrow \; M_{12} = +\infty, \; M_{13} = M_{23} \equiv \mathcal{M}_3; \; M_1 = M_2 = M_3 = \widetilde{M_3}$$
(4.5)

where the eqs. (3.11) and (3.6) have been used, respectively, to define the parameters $\mathcal{M}_j$ and $\widetilde{M_j}$ in the right hand sides of the relationships above.

In appendix A.5 we record the kink type solution [see eq. (A.49)] which corresponds to the GSG reduced model called double sine-Gordon theory. This solution corresponds to baryon number $4N_C$ [see eq. (A.53)]. Thus, from (A.46), (3.11) and (3.6) one has

$$\Phi_1 = \Phi_2 = \frac{1}{2}\Phi_3 = \frac{1}{2}\phi \Rightarrow M_{12} = +\infty, \; M_{13} = M_{23} \equiv \mathcal{M}_K; \; M_1 = M_2 \equiv \mathcal{M}_K, \; M_3 \equiv \mathcal{M}_{2K} \,. \quad (4.6)$$

The solutions with baryon numbers $2N_C$ and $3N_C$ correspond to composite configurations formed by multi-solitons of the GSG model. These states (i.e. multi-baryons) deserve a careful treatment which we hope to undertake in future.

## 4.1 GSG solitons and the states with baryon number $N_C$

For the particular cases (4.3)–(4.5) one can rewrite the action (4.1) such that for each case the terms quadratic in time derivatives reduce to a term depending only on one variable, say $z_i^{(\hat{l})}$, related to the $\hat{l}$'th column of the matrix $A$. The reason is that the symmetries of the quantum mechanical lagrangian and actual manifold on which $A(t)$ lives depend on the properties of the ansatz $g_0$. For the ansatz $g_0$ related to the GSG model one can see that the space-time dependent field $g$ in eq. (3.1) can be rewritten only in terms of certain columns of $A$. For example, in the case (4.5) above the matrix $g(x,t)$ can be written as

$$g_{\alpha\beta}(x,t) = [Ag_0A^{-1}]_{\alpha\beta}$$
$$= \delta_{\alpha\beta}e^{i\beta_0\hat{\varphi}} - 2i\sin(\beta_0\hat{\varphi})z_\alpha^{(3)}z_\beta^{(3)\,\star}, \quad (4.7)$$

which clearly depends only on the third column of $A$. So, we may think the left hand side of (3.2), i.e. $[\widetilde{S}(g(x,t)) - \widetilde{S}(g_0(x))]$, entering the expression of the semi-classical quantization approach, would in principle be written only in terms of the third column of

$A$. However, in order to envisage certain local symmetries it is useful to write the terms first order in time derivatives as depending on the full parameters $z_i^{(j)}$ of the field $A$. These terms arise from the WZW term and provides the Gauss law type $N_z$ number conservation law [See eq. (4.14) below]. An additional $\mathrm{SU}(2) \in H$ (see (4.8)) local symmetry will be described below. Moreover, this picture is in accordance with the counting of the degrees of freedom. In fact, the effective action (3.2) possesses the local gauge symmetry (3.4), where in the case of field configuration (4.5) the gauge group $H$ becomes

$$H = \mathrm{SU}(2) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_Y, \tag{4.8}$$

with the last two $\mathrm{U}(1)$ factors related to baryon number and hypercharge, respectively. Thus, the effective action (4.1) will be an action for the coordination describing the coset space $G/H = \mathrm{SU}(3) \times \mathrm{U}(1)_B / \mathrm{SU}(2) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_Y = CP^2$. The $\Phi_i$ fields and symmetries of $g_0$ also determine the values and relationships between the parameters $M_{ij}$ in (4.3)–(4.5), such that certain coefficients in (4.1) depending on these parameters vanish identically, thus leaving a subset of $z_i^{(j)}$ variables which must be consistent with the counting of the degrees of freedom. For example this picture is illustrated in the case (4.5) where the coefficients $(M_{12}^{-1} + M_{13}^{-1} - M_{23}^{-1})$ and $(M_{12}^{-1} - M_{13}^{-1} + M_{23}^{-1})$ vanish identically, leaving an action with kinetic term depending only on the variables $z_\alpha^{(3)}$. However, the mass and WZW terms are conveniently written in terms of the complete $z_i^{(j)}$ variables.

So, for each case in (4.3)–(4.5) labelled by $\hat{l}$, the action can be written as

$$S[z_p^{(i)}(t)] = \frac{1}{2} \int dt\, \mathcal{M}_{\hat{l}}^{-1} \Big[ \dot{z}_\alpha^{(\hat{l})} \dot{z}_\alpha^{(\hat{l})\star} - \dot{z}_\alpha^{(\hat{l})} z_\alpha^{(\hat{l})\star} z_\beta^{(\hat{l})} \dot{z}_\beta^{(\hat{l})\star} \Big] - $$
$$i\frac{N_C}{2} \int dt \sum_{i,p} n_i \Big[ \dot{z}_p^{(i)} z_p^{(i)\star} - z_p^{(i)} \dot{z}_p^{(i)\star} \Big] - \frac{2\pi}{N_c \widetilde{M}_{\hat{l}}} \int dt \sum_{i,j} \widetilde{m}_i^2 |z_i^{(j)}|^2. \tag{4.9}$$

In the relation above we must assign the relevant set of values to the indices $n_i$ ($i = 1, 2, 3$) (see appendix) for the relevant case in (4.3)–(4.5). The first term in (4.9) is the usual $CP^2$ quantum mechanical action, while the terms first order in time-derivatives are modifications due to the WZ term, as arisen from (3.2) and (3.10). Notice that the last term was originated from the unequal quark mass terms.

Following similar steps as in the single baryon case (see eqs. (3.23)–(3.24)) one can obtain the hamiltonian

$$H = \frac{1}{2\mathcal{M}_{\hat{l}}} \Big( Dz^{(\hat{l})} \Big)_\alpha^\dagger \Big( Dz^{(\hat{l})} \Big)_\alpha + \frac{2\pi}{N_c \widetilde{M}_{\hat{l}}} \sum_{i,j} \widetilde{m}_i^2 |z_i^{(j)}|^2, \tag{4.10}$$

where $\Big( Dz^{(\hat{l})} \Big)_\alpha = \dot{z}_\alpha^{(\hat{l})} - z_\alpha^{(\hat{l})} (z_\beta^{(\hat{l})\star} \dot{z}_\beta^{(\hat{l})})$.

Similarly, the corresponding second Casimir becomes

$$Q^A Q_A = \frac{1}{2} Q_{\alpha\beta} Q_{\beta\alpha},$$
$$= \frac{1}{2} N_C^2 \sum_i |n_i|^2 + \frac{1}{4\mathcal{M}_{\hat{l}}^2} \Big( Dz^{(\hat{l})} \Big)_\alpha^\dagger \Big( Dz^{(\hat{l})} \Big)_\alpha \tag{4.11}$$

Then from (4.10)–(4.11) and taking into account $Q^A Q_A = C_2 + \frac{1}{2N_f} \sum_i (\mathcal{Q}_B^i)^2$ one can get

$$H = 2\mathcal{M}_{\hat{l}} \Big( C_2 + \frac{1}{2N_f} \sum_i (\mathcal{Q}_B^i)^2 - \frac{1}{2} N_C^2 \sum_i |n_i|^2 \Big) + \frac{2\pi}{N_c \widetilde{M}_{\hat{l}}} \sum_{i,j=1}^{3} \widetilde{m}_i^2 |z_i^{(j)}|^2, \quad (4.12)$$

where $\mathcal{Q}_B^i = n_i N_C$ for a convenient choice of the indices $n_i$, which in the cases (4.3)–(4.5) is simply $|n_i| = 1$ [see also eqs. (A.19), (A.26) and (A.33) for 1-soliton configurations]. The parameters $\mathcal{M}_{\hat{l}}$, $\widetilde{M}_{\hat{l}}$ can be computed for the relevant solitons. They become

$$\frac{1}{2\mathcal{M}_{\hat{l}}} = \frac{1}{\widetilde{m}} \frac{2\sqrt{2}}{3} \Big( \frac{N_C}{\pi} \Big)^{3/2}, \qquad \frac{1}{2\widetilde{M}_{\hat{l}}} = \frac{1}{\sqrt{2}\,\widetilde{m}} \Big( \frac{N_C}{\pi} \Big)^{3/2} \quad (4.13)$$

Some comments concerning the two hamiltonians (3.30) and (4.12) are in order here. Even though they correspond to one baryon state (baryon number $N_C$) they look different. In fact, the hamiltonian (4.12) incorporates additional terms. First, due to the ansatz (2.6) related to the GSG model one has some set of field solutions comprising in total three possibilities (4.3)–(4.5) with baryon number $N_C$, each case being characterized by the set of parameters $\mathcal{M}_{\hat{l}}$, $\widetilde{M}_{\hat{l}}$ and relevant combinations of the indices $n_j$ which are related to the baryon number of the configuration $\{\Phi_j\}$, $j = 1, 2, 3$. So, the terms $-\frac{N_C^2}{2}$ and $\frac{N_C^2}{2N_f}$ in (3.30) translate to $-\frac{N_C^2}{2} \sum_i n_i^2$ and $\frac{1}{2N_f} \sum_i (\mathcal{Q}_B^i)^2$, respectively, in the new hamiltonian (4.12). Second, the mass term expression allows an exact summation due to unitarity, thus giving a constant additional term to the hamiltonian (see below). The corresponding term in (3.30), obtained in [6], does not permit an exact summation.

## 4.2 Lowest lying baryon state and the GSG soliton

So far, the treatment for each case (4.3)–(4.5) followed similar steps; however, in order to compute the quantum correction to the soliton mass we choose the one from the classification (4.3)–(4.5) with the minimum classical energy solution. Thus, taking into account the "physically" motivated inequalities $m_3 < m_1 < m_2$ ( or $\mu_3 < \mu_1 < \mu_2$) [eq. (A.51) relates the $\mu_j$'s and the $m_j$'s] one observes that the soliton with mass $M_2^{sol}$ [see eq. (A.23)] possesses the smallest mass according to the relationship (A.34). This corresponds to the *second case* (4.4) classified above; so one must set the index $\hat{l} = 1$ in the action (4.9).

The variation of the action (4.9) under $z_\alpha^{(j)} \to e^{i\delta(t)} z_\alpha^{(j)}$ is due to the WZW term: $\Delta S = N_c (n_1 + n_2 + n_3) \int dt\, \dot{\delta}$. This implies

$$N_z = \frac{\Delta S}{\Delta \dot{\delta}} = N_c \Big( n_1 + n_2 + n_3 \Big), \quad (4.14)$$

which is an analog of the Gauss law, and restricts the allowed physical states [28]. For the soliton configuration with baryon number $N_C$, (4.4), under consideration in this subsection, we have $n_1 = -n_2 = -n_3 = -1 \to n_1 + n_2 + n_3 = 1$ [see eq. (A.25) ] implying

$$N_z = N_C. \quad (4.15)$$

Therefore, for any wave function, written as a polynomial in $z$ and $z^\star$ the number of the $z$ minus the number of the $z^\star$ must be equal to $N_C$. But due to a larger local symmetry we will have more restrictions. Thus, as commented earlier the (massless part) effective action (4.9) is invariant under the local SU(2) symmetry. This can be easily seen by defining "local gauge potentials"

$$\widetilde{A}_{\beta\alpha}(t) = -\sum_p z_p^{(\beta)\star} \dot{z}_p^{(\alpha)}, \quad \alpha, \beta = 2, 3. \tag{4.16}$$

Under the local gauge transformation corresponding to $\Lambda(t)$, one has

$$\widetilde{A}(t) \rightarrow e^{i\Lambda} \widetilde{A} e^{-i\Lambda} + \partial_t e^{i\Lambda} e^{-i\Lambda}. \tag{4.17}$$

Then we have that the WZW term in (4.9) for the variables $z_p^\alpha$, $\alpha = 2, 3$ (take $\hat{l} = 1$, $n_2 = n_3 = 1$) remain invariant under the transformation (4.17)

$$iN_C \int dt \, \mathrm{Tr} \, \dot{z}_p^{(\alpha)\star} z_p^{(\beta)} \equiv iN_C \int dt \, \mathrm{Tr} \, \widetilde{A} \ \Rightarrow \ iN_C \int dt \, \mathrm{Tr} \widetilde{A} \tag{4.18}$$

Remember that the variables $z_p^\alpha$ do not appear in the kinetic term of (4.9). The local symmetry above imply that the allowed physical states must be singlets under the SU(2) symmetry in flavor space. So, the wave functions for $z's$ only (analogous to quarks only for QCD) must be of the form

$$\psi_2(z) = \Pi_{i=1}^{N_C}\left(\epsilon_{\alpha_1\alpha_2} z_{i_1}^{(\alpha_1)} z_{i_2}^{(\alpha_2)}\right), \quad \alpha_1, \alpha_2 = 2, 3, \tag{4.19}$$

where $1 \leq i_1, i_2 \leq N_f$.

Then, taking into account the restrictions of the types (4.15) and (4.19) the most general state can be written as

$$\widetilde{\psi}(z, z^\star) = \psi_2(z)\left[\Pi_{\{p,q\}}(z_p^{(\alpha)\star} z_q^{(\alpha)})^{n_{pq}}\right], \tag{4.20}$$

and the products are defined for some sets of indices. This wave function generalizes the one given in (3.27).

Next, let us compute the mass of the state represented for wave functions of the form $\widetilde{\psi}(t) = \psi_2(z) \Pi_i(z_i^{(1)})^{p_i}$ where $(\sum_{i=1}^{N_f} p_i = N_C)$.

Combining the hamiltonian (4.12), the relevant parameters (4.13) and the classical soliton mass term, for the $R$ baryon we have

$$M(baryon) = M_{\mathrm{classical}}\left\{1 + \frac{3}{4}(\frac{\pi}{2N_C})^2\left[C_2(R) - \frac{N_C^2}{2}(N_f - 1) + \frac{1}{2\widetilde{m}^2}\sum_i \widetilde{m}_i^2\right]\right\} \tag{4.21}$$

where

$$M_{\mathrm{classical}} = 4\widetilde{m}\left(\frac{2N_C}{\pi}\right)^{1/2}, \quad \widetilde{m}^2 = \frac{1}{13}\left(\frac{m^2}{m_0}\right)\left(6m_1 + 3m_2\right). \tag{4.22}$$

The last term in (4.12) simplifies to a constant term by unitarity condition of the matrix elements $z_i^{(j)}$ and the parameter $\widetilde{m}$ corresponds to the one-soliton parameter once the identification $\gamma_2^2 = 2\beta_0^2 \widetilde{m}^2$ is made in (A.22) by comparing the SG one-solitons (1.1) and (A.20). Even though the computations are explicitly made for $N_f = 3$ it is instructive to leave the number of flavors as a variable. In the case of the **10** baryon one has

$$M(baryon) = M_{\text{classical}}\left[1 + \frac{3\pi^2}{32}\frac{N_f - 1}{N_C} - \frac{3\pi^2}{32}\frac{(N_f - 1)^2}{N_f} + \frac{3}{2}\right]. \qquad (4.23)$$

In the following we discuss the correction terms to the earlier expression (3.31) for the **10** baryon as compared to the last improved expression (4.23). The quantum correction of (3.31) is multiplied by 3/4 and the last two terms in (4.23) are new contributions due to the GSG ansatz used and the unequal quark mass terms. The last term contribution in (4.21) was simplified providing a numerical term 3/2 in (4.23) thanks to unitarity and the relationship between the quark masses (A.22) which is a condition to get the relevant soliton solution. This term apparently may not be consistent with a quantum correction around the classical solution since consistency with the semi-classical approximation requires it be small compared to one. However, this term must be combined with the third term which gives a negative value contribution and is an additional term independent of $N_C$, as is the last numerical 3/2 term under discussion. In fact, for $N_C = 3$, $N_f = 3$, numerically these two terms contribute $\sim 0.27$, which is acceptable. The $N_C$ dependent term numerically becomes $\sim 0.62$ (the term 0.82 of (3.31) has been multiplied by 3/4). Adding all the quantum contributions one has 0.89, which increases the earlier numerical value 0.86 of (3.31) in $\sim 3.5\%$. In fact, this is a small correction to the already known value which was obtained using the ansatz (2.8) in [4, 5].

## 4.3 Possible vibrational modes and the GSG model

The only static soliton configurations with baryon number $N_C$, which emerge in the strong-coupling regime of $\text{QCD}_2$, are the ones we have considered above in eqs. (4.3)–(4.5). Precisely, these are the one-solitons of the GSG model which, in subsection 4.2, have been the subject of semi-classical treatment. Their quantum corrections by time-dependent rotations in flavor space have been computed, we focused on the one with the lowest classical mass. Since in two dimensions there are no spin degrees of freedom, in order to search for higher excitations we must look for vibrational modes which might in principle exist. These type of excitations in the strong coupling limit can be found as classical time-dependent solutions of the GSG equations of motion (A.9)–(A.10). Looking at time-dependent solutions of type (4.4) [see eq. A.20] one has that the field $\varphi_2$ satisfies ordinary sine-Gordon equation

$$\partial_{tt}\varphi_2 - \partial_{xx}\varphi_2 + 2\widetilde{m}^2\sqrt{\frac{4\pi}{N_C}}\sin\left(\frac{4\pi}{N_C}\varphi_2\right), \quad \varphi_1(x,t) \equiv 0. \qquad (4.24)$$

The time dependent one-soliton solution of (4.24) for the field $\varphi_2$, determines the configuration $\{\Phi_1, \Phi_2, \Phi_3\}$ in (4.4) with baryon number $N_C$ in the $\text{QCD}_2$ context. To look for higher excitations, for example, one can search for a coupled state of one-baryon and

*breather* type vibrations (soliton-antisoliton bound states) of the GSG system, which can give a total baryon number $N_C$. We were not be able to find a more general time-dependent mixed single-baryon plus vibrational state with baryon number $N_C$ for the general GSG equation. For example, this type of solution, if it exists, may be useful in order to study meson-baryon scattering as considered in [7]. As it is well known the SG eq. (4.24) does give vibrational solutions in the form of breather states (meson states), for later use we simply recall that in the large $N_C$ limit the lowest-lying mesons have masses of order $\sqrt{m_q e_c}$ [29] ($m_q$ is defined in eq. (5.5) below). We refer the reader to ref. [5] for more discussion, such as the various meson couplings to baryons with different degrees of exoticity.

## 5. The GSG solitons and the exotic baryons

### 5.1 The first exotic baryon

Here we will follow the analog of the rigid-rotor approach (RRA) to quantize solitons and obtain exotic states. In this method it is assumed that the higher order representation multiplets are different rotational (in spin and isospin) states of the same object (the "classical baryon", i.e the soliton field) [23]. This assumption has allowed in the past the obtention of some relations between the characteristics of the nonexotic baryon multiplets which are satisfied up to a few percent in nature. However, see refs. [30, 31] for some critiques to this conventional approach for exotic baryons. According to these authors the conventional RRA, in which the collective rotational approach and vibrational modes of the soliton are assumed to be decoupled, and only the rotational modes are quantized, is only justified at large $N_C$ for nonexotic collective states in SU(3) models. On the other hand, the bound state approach (BSA) to quantize solitons, due to Callan-Klebanov [31], considers broken SU(3) symmetry in which the excitations carrying strangeness are taken as vibrational modes, and should be quantized as harmonic vibrations. However, for exotic states the Callan-Klebanov approach does not reproduce the RRA result; indeed this approach gives no exotic resonant states when applied to the original Skyrme model [31]. There was intensive discussion of connections between the both approaches mentioned above. The rotation-vibration approach (RVA) (see [32] and references therein) includes both rotational (zero modes) and vibrational degrees of freedom of solitons and is a generalization of the both methods above, which therefore appear in some regions of the RVA method when certain degrees of freedom are frozen. A major result of the RVA method is that pentaquark states *do* indeed emerge in both methods above, i.e. in the RRA and BSA. In order to illustrate the present situation of the theoretical controversy let us mention that the RVA approach was criticized in [33], and the reply to this criticism was given in [34].

Following the analog of the RRA, the expression (4.21) can be used to compute the energy of the first exotic baryon $\mathcal{E}_1$ (a state containing $N_C + 1$ quarks and one antiquark) by taking the corresponding Casimir $C_2(\mathcal{E}_1)$ for $R = \mathbf{35}$ of flavor relevant to the exotic state in two-dimensions. This state is an analogue of the $\overline{\mathbf{10}}, \mathbf{27}$ and $\mathbf{35}$ states in four dimensions. So, following [5], in the conventional RRA one has that the mass of the first

exotic state becomes

$$M(\mathcal{E}_1) = M(classical)\left\{1 + \frac{3}{4}\left[\frac{\pi^2}{8}\frac{1}{N_C}\left(3 + N_f - \frac{6}{N_f}\right) + \frac{3\pi^2}{8}\frac{1}{N_C^2}\left(N_f - \frac{3}{N_f}\right)\right]\right.$$
$$\left. - \frac{3\pi^2}{32}\frac{(N_f-1)^2}{N_f} + \frac{3}{2}\right\} \quad (5.1)$$

In the interesting case $N_C = 3$, $N_f = 3$ this becomes

$$M(\mathbf{35}) = M(classical)\left\{1 + \frac{3}{4}\frac{\pi^2}{4} - \frac{\pi^2}{8} + \frac{3}{2}\right\}. \quad (5.2)$$

In this case the correction due to quantum fluctuations around the classical solution is still larger than the classical term, as it was in the earlier computation (3.33). However, numerically in eq. (5.2) the correction is 2.12, whereas in eq. (3.33) it was 2.46. In fact, the contribution in (5.2) decreases in 0.34 units the earlier computation. So, we may claim that the introduction of unequal quark masses and the ansatz given by the GSG model slightly improve the semi-classical approximation.

Moreover, notice that the ratio of the experimental masses of the $\Theta^+(1530)$ and the nucleon is 1.63. On the other hand, the ratio of the first exotic to that of the lightest baryon in the $QCD_2$ model becomes

$$\frac{M_{\mathbf{35}}}{M_{\mathbf{10}}} = \frac{1 + \frac{3\pi^2}{16} - \frac{\pi^2}{8} + \frac{3}{2}}{1 + \frac{\pi^2}{16} - \frac{\pi^2}{8} + \frac{3}{2}} \sim 1.65, \quad (5.3)$$

which is only 1% larger to its 4D analog. This must be compared to the earlier calculation which gave a value 17% larger [see eq. (3.33)]. However, the result in (5.3) could be a numerical coincidence, since in two dimensions we are not considering the spin degrees of freedom that is important in $QCD_4$, even though the effects of unequal quark masses $m_3 < m_1 < m_2$ have been incorporated as an exact (without using perturbation theory) contribution to the hamiltonian.

## 5.2 Exotic baryon higher multiplets

Let us consider exotic states $\mathcal{E}_p$ containing $p$ antiquarks and $N_C + p$ quarks. In the case $N_C = 3$, $N_f = 3$, the only allowed $\mathcal{E}_2$ state is a $\mathbf{81}$ representation of flavor. In the particular case $N_f = 3$, for general $N_C$ the mass of the $\mathcal{E}_p$ state is

$$M(\mathcal{E}_p) = M(classical)\left\{1 + \frac{3}{4}\left(\frac{\pi}{2N_C}\right)^2\left[N_C(p+1) + p(p+2) - \frac{2}{3}N_C^2\right] + 3/2\right\}, \quad (5.4)$$

where the correction is considerably larger than unity. For example for $N_C = 3$ the mass correction becomes 3.76 units. Even though this correction is one unit less than the one obtained in [5], we would not consider it as a consistent semi-classical approximation for $N_C = 3$. However, we may consider the spacing $\Delta$ between $\mathcal{E}_{p+1}$ and $\mathcal{E}_p$ exotic states, which for large $N_C$ becomes

$$\Delta \equiv \mathcal{E}_{p+1} - \mathcal{E}_p = \left(\frac{3}{4}\right)\frac{\pi^2}{4}\frac{M_{\text{classical}}}{N_C} \sim 3.8\sqrt{e_c m_q}\,; \qquad m_q \equiv \frac{2m_1 + m_2}{3} \quad (5.5)$$

so, the constant $\Delta$ of [5] is decreased by a factor of 3/4. Since $M_{\text{classical}}$ is $\mathcal{O}(N_C^1)$, then the parameter $\Delta$ is a constant $\mathcal{O}(N_C^0)$ as the exoticity $p$ is increased. Notice that the low-lying mesons masses are $\mathcal{O}(N_C^0)$ in the large $N_C$ limit [5]. This would mean that the constant $\Delta$ value is like the addition of a meson to the $p-$state, in the form of quark-antiquark pair, in order to progress to the next excitation $p+1$ [35]. Remember that the low-lying mesons in the SG theory have masses $\sim 3.2\sqrt{m_q e_c}$ [29], which are very close to the spacing $\Delta$ defined in (5.5).

## 5.3 Radius parameter of the QCD$_2$ exotic baryons

In QCD$_2$, as found above, the quantum correction to the mass depends on one analogue of the moment of inertia appearing in four dimensions. Following [5] one considers

$$I = M(classical) < r^2 >, \qquad (5.6)$$

the effective soliton radius can be defined by

$$\ll r \gg \equiv \sqrt{< r^2 >}. \qquad (5.7)$$

Let us compare the quantum mass formula (5.4) with the corresponding relation in four dimensions [23] in the large $N_C$ limit $(N_C \gg p \gg 1)$, so one has

$$I = \frac{8N_C^2}{3\pi^2 M_{\text{classical}}}, \qquad (5.8)$$

and then

$$\ll r \gg = \sqrt{\frac{I}{M_{\text{classical}}}} = \sqrt{\frac{8}{3}}\frac{N}{\pi M_{\text{classical}}} = \frac{1}{0.96\pi N_f^{1/4}\sqrt{e_c m_q}}, \qquad (5.9)$$

where $m_q$ was defined in (5.5). For $N_f = 3$ flavors, $e_c = 100\, MeV$ for the coupling, and quark masses $m_3 = 4\,\text{MeV}$, $m_1 = 54.5\,\text{MeV}$, and $m_2 = 55.1\text{MeV}$ [these values satisfy the relationship $13m_3 = 5m_1 - 4m_2$ relevant in two-dimensions as is obtained from (A.22) and (A.51)], we get for the effective baryon radius $\approx 1/(294 MeV) \sim 0.7\,\text{fm}$. This is 12.5% less than the radius estimated in [5] for QCD$_2$ exotic baryons. As a curiosity, notice that the radius parameter of $\Theta^+$ has been estimated to be around $1.13\,\text{fm} = 5.65\,\text{GeV}^{-1}$ (see e.g. [36] and references therein).

## 6. Discussion

We have extended the results of refs. [4, 5] concerning several properties of normal and exotic baryons by including unequal quark mass parameters. In the case of $N_f = 3$ flavors, the low-energy hadron states are described by the $su(3)$ generalized sine-Gordon model, providing a framework for the exact computations of the lowest-order quantum corrections of various quantities, such as the masses of the normal and exotic baryons. The semi-classical quantization method we adopted is an analogue of the rigid-rotor approach (RRA)

applied in four dimensional QCD to quantize normal and exotic baryons (see e.g. [23]). Even though there is no spin in 2D, we have compared our results to their analogues in 4D; so, obtaining various similarities to the results from the chiral-soliton approaches in $QCD_4$. The RRA we have followed, as discussed in section 5, may be justified in our case since there is no mixing between the intrinsic vibrational modes and the collective rotation in flavor space degrees of freedom [30]. It is remarkable that the GSG ansatz (2.6), with soliton solutions which take into account the unequal quark mass parameters, allowed us to improve the lowest order quantum corrections for various physical quantities, such as the baryon masses; in this way rendering the semi-classical method more reliable in the large $N_C$ limit.

Other properties of the baryons such as a proper treatment of $k-$baryon bound states (extending the results of [37] for GSG type ansatz), including baryon-meson scattering amplitudes, are still to be addressed in the future.

We have found that the remarkable double sine Gordon model arises as a reduced GSG model bearing a kink($K$) type solution describing a multi-baryon; so, the description of some resonances in $QCD_2$ may take advantage of the properties of the $K\bar{K}$ system which are being considered in the current literature [38 − 40].

## Acknowledgments

## A. $sl(3,\mathbb{C})$ GSG model, soliton and kink solutions

Here we summarize some properties of the $sl(3,\mathbb{C})$ GSG model [21, 22] relevant to our discussions above, such as the soliton and kink spectrum. The discussions make some connection to the $QCD_2$ developments above, such as (multi-) baryon number of solitons and kinks. The third soliton solution with baryon number $N_C$ is new.

The generalized sine-Gordon model (GSG) related to $sl(3,\mathbb{C})$ is defined by [19, 20, 22]

$$S = \int d^2x \sum_{i=1}^{3} \left[ \frac{1}{2}(\partial_\mu \Phi_i)^2 + \mu_i \Big(\cos\beta_0 \Phi_i - 1\Big) \right]. \tag{A.1}$$

Since in this case one has two simple roots there are two independent real fields, $\varphi_{1,2}$, such that

$$\Phi_1 = \nu_1(2\varphi_1 - \varphi_2); \;\; \Phi_2 = \nu_2(2\varphi_2 - \varphi_1); \;\; \Phi_3 = \nu_3(r\,\varphi_1 + s\,\varphi_2), \tag{A.2}$$
$$\nu_i\,(i=1,2,3),\, s,\, r \in \mathbb{R}$$

which must satisfy the constraint

$$\Phi_3 = \delta_1 \Phi_1 + \delta_2 \Phi_2, \tag{A.3}$$

where $\delta_1, \delta_2$ are some real numbers. The $\Phi$ fields dependence on the $\varphi'$s can be explained in the context of the Lie algebraic construction of the classical version of the model [20, 13].

Taking into account (A.2)–(A.3) and the fact that the fields $\varphi_1$ and $\varphi_2$ are independent we may get the relationships

$$\nu_2\delta_2 = \rho_0\nu_1\delta_1 \quad \nu_3 = \frac{1}{r+s}(\nu_1\delta_1 + \nu_2\delta_2); \quad \rho_0 \equiv \frac{2s+r}{2r+s} \tag{A.4}$$

The $sl(3,\mathbb{C})$ model has a potential density

$$V[\varphi_i] = \sum_{i=1}^{3} \mu_i\Big(1 - \cos\beta_i\Phi_i\Big) \tag{A.5}$$

In the $sl(3,\mathbb{C})$ construction of [22] the parameters $\delta_i$ depend on the couplings $\beta_i$ $[\beta_i \equiv \beta_0\nu_i]$ and they satisfy certain relationship. This is obtained by assuming $\mu_i > 0$ and the zero of the potential given for $\Phi_i = \frac{2\pi}{\beta_0}n_i$, which substituted into (A.3) provides

$$n_1\delta_1 + n_2\delta_2 = n_3, \quad n_i \in \mathbb{Z} \tag{A.6}$$

The last relation combined with (A.4) gives

$$(2r+s)\frac{n_1}{\nu_1} + (2s+r)\frac{n_2}{\nu_2} = 3\frac{n_3}{\nu_3}. \tag{A.7}$$

The periodicity of the potential implies an infinitely degenerate ground state and then the theory supports topologically charged excitations. So, consider the vacuum lattice defined by

$$(\Phi_1, \Phi_2) = \frac{2\pi}{\beta_0}(n_1, n_2), \quad n_a \in \mathbb{Z}. \tag{A.8}$$

It is convenient to write the equations of motion in terms of the independent fields $\varphi_1$ and $\varphi_2$

$$\partial^2\varphi_1 = -\mu_1\beta_1\Delta_{11}\sin[\beta_1(2\varphi_1 - \varphi_2)] - \mu_2\beta_2\Delta_{12}\sin[\beta_2(2\varphi_2 - \varphi_1)] +$$
$$\mu_3\beta_3\Delta_{13}\sin[\beta_3(r\varphi_1 + s\varphi_2)] \tag{A.9}$$
$$\partial^2\varphi_2 = -\mu_1\beta_1\Delta_{21}\sin[\beta_1(2\varphi_1 - \varphi_2)] - \mu_2\beta_2\Delta_{22}\sin[\beta_2(2\varphi_2 - \varphi_1)] +$$
$$\mu_3\beta_3\Delta_{23}\sin[\beta_3(r\varphi_1 + s\varphi_2)], \tag{A.10}$$

where the $\Delta'_{ij}s$ depend on $\beta_0$, $\nu_j$ $(j=1,2,3)$, $r$, $s$, $\delta_a$ $(a=1,2)$.

Notice that the eqs. of motion (A.9)–(A.10) exhibit the symmetries

$$\varphi_1 \leftrightarrow \varphi_2, \quad \mu_1 \leftrightarrow \mu_2, \quad \nu_1 \leftrightarrow \nu_2, \quad \delta_1 \leftrightarrow \delta_2, \quad r \leftrightarrow s; \tag{A.11}$$
$$\text{and} \quad \varphi_a \leftrightarrow -\varphi_a, \quad a = 1, 2 \tag{A.12}$$

In the following we write the 1-soliton(antisoliton), 1-kink(antikink) and bounce type solutions and compute the relevant (multi-)baryon numbers associated to the U(1) symmetry in the context of QCD$_2$.

## A.1 One soliton/antisoliton pair associated to $\varphi_1$

The functions

$$\varphi_1 = \frac{4}{\beta_0}\arctan\{d\,\exp[\gamma_1(x - vt)]\}, \quad \varphi_2 = 0, \tag{A.13}$$

satisfy the system of equations (A.9)–(A.10) for the set of parameters

$$\nu_1 = 1/2, \; \delta_1 = 2, \; \delta_2 = 1, \; \nu_2 = 1, \; \nu_3 = 1, \; r = 1. \tag{A.14}$$

provided that

$$13\mu_3 = 5\mu_2 - 4\mu_1, \quad \gamma_1^2 = \frac{\beta_0^2}{13}(6\mu_2 + 3\mu_1). \tag{A.15}$$

This solution is precisely the sine-Gordon 1-soliton associated to the field $\varphi_1$ with mass

$$M_1^{sol} = \frac{8\gamma_1}{\beta_0^2}. \tag{A.16}$$

From (A.2) and taking into account the parameters (A.14) one has the relationships between the GSG fields

$$\Phi_1 = -\Phi_2 = \Phi_3 = \varphi_1 \tag{A.17}$$

Moreover, from (A.6)–(A.7) and (A.14) one gets the relationships

$$n_1 = -n_2 = n_3 \tag{A.18}$$

Taking into account the QCD$_2$ motivated formula (2.17) and (A.18) one can compute the baryon number of the GSG soliton (A.13) taking $n_1 = 1$

$$\mathcal{Q}_B^{(1)} = N_C, \tag{A.19}$$

where the superindex (1) refers to the associated $\varphi_1$ field nontrivial solution.

## A.2 One soliton/antisoliton pair associated to $\varphi_2$

The functions

$$\varphi_2 = \frac{4}{\beta_0}\arctan\{d\exp[\gamma_2(x - vt)]\}, \quad \varphi_1 = 0 \tag{A.20}$$

solve the system (A.9)–(A.10) for the choice of parameters

$$\nu_1 = 1, \; \delta_1 = 1, \; \delta_2 = 2, \; \nu_2 = 1/2, \; \nu_3 = 1, \; s = 1 \tag{A.21}$$

provided that

$$13\mu_3 = 5\mu_1 - 4\mu_2, \quad \gamma_2^2 = \frac{\beta_0^2}{13}(6\mu_1 + 3\mu_2). \tag{A.22}$$

This is the sine-Gordon 1-soliton associated to the field $\varphi_2$ with mass

$$M_2^{sol} = \frac{8\gamma_2}{\beta_0^2}. \tag{A.23}$$

As above from (A.2) and the set of parameters (A.21) one has the relationships

$$-\Phi_1 = \Phi_2 = \Phi_3 = \varphi_2. \tag{A.24}$$

From (A.6)–(A.7) and (A.21) one gets the relationship

$$n_1 = -n_2 = -n_3. \tag{A.25}$$

So, taking into account the QCD$_2$ motivated formula (2.17) and (A.25) one computes the baryon number of this GSG soliton taking $n_2 = 1$

$$\mathcal{Q}_B^{(2)} = N_C, \tag{A.26}$$

where the superindex (2) refers to the associated $\varphi_2$ field.

### A.3 1-soliton/1-antisoliton pairs associated to $\hat{\varphi} \equiv \varphi_1 = \varphi_2$

In the case $\varphi_1 = \varphi_2$ one has the 1-soliton solution $\hat{\varphi}$ of the system (A.9)–(A.10) associated to the parameters

$$\nu_1 = 1, \ \delta_1 = -1/2, \ \nu_2 = 1, \ \delta_2 = -1/2, \ \nu_3 = -1/2, \ r = s = 1. \tag{A.27}$$

One has the 1-soliton

$$\begin{aligned}
\varphi_1 &= \varphi_2 \equiv \hat{\varphi}, \\
\hat{\varphi} &= \frac{4}{\beta_0} \arctan\{d \ \exp[\gamma_3(x - vt)]\},
\end{aligned} \tag{A.28}$$

which requires

$$\gamma_3^2 = \beta_0^2 \Big(\mu_1 + \frac{1}{2}\mu_3\Big), \quad \mu_1 = \mu_2. \tag{A.29}$$

This is a sine-Gordon 1-soliton associated to both fields $\varphi_{1,2}$ in the particular case when they are equal to each other. It possesses a mass

$$M_3^{sol} = \frac{8\gamma_3}{\beta_0^2}. \tag{A.30}$$

In view of the symmetry (A.11) which are satisfied by the parameters (A.27) and (A.29) one can think of this solution as doubly degenerated.

As above, from (A.2) and the set of parameters (A.27) one has the following relationships

$$\Phi_1 = \Phi_2 = -\Phi_3 = \hat{\varphi}. \tag{A.31}$$

From (A.6)–(A.7) and (A.27) one gets the relationship

$$-2n_3 = n_1 + n_2. \qquad (A.32)$$

So, taking into account the QCD$_2$ motivated formula (2.17) and (A.32) one computes the baryon number of this GSG solution taking $n_3 = -1$

$$\mathcal{Q}_B^{(\hat{\varphi})} = N_C, \qquad (A.33)$$

where the superindex refers to the associated $\hat{\varphi}$ field.

### A.3.1 Antisolitons and general N-solitons

The GSG system (A.9)–(A.10) reduces to the usual SG equation for each choice of the parameters (A.14), (A.21) and (A.27), respectively. Then, the $N-$soliton solutions in each case can be constructed as in the ordinary sine-Gordon model.

Using the symmetry (A.12) one can be able to construct the *1-antisolitons* corresponding to the soliton solutions (A.13), (A.20) and (A.28) simply by changing their signs $\varphi_a \rightarrow -\varphi_a$.

### A.4 Mass splitting of solitons

It is interesting to write some relationships among the various soliton masses.

i) For $\mu_1 \neq \mu_2$ one has respectively the two 1-solitons, (A.13) and (A.20), with masses (A.16) and (A.23) related by

$$(M_1^{sol})^2 - (M_2^{sol})^2 = \frac{48 N_C}{\pi}(\mu_2 - \mu_1). \qquad (A.34)$$

ii) For $\mu_1 = \mu_2$, there appears the third soliton solution (A.28)–(A.29). Then, taking into account (A.15), (A.22), (A.29), (A.34) and the third soliton mass (A.30) we have the relationships

$$M_1^{sol} = M_2^{sol}, \qquad M_3^{sol} = \sqrt{3/2}\, M_1^{sol}, \qquad (A.35)$$

$$\gamma_1 = \gamma_2 = \sqrt{2/3}\,\gamma_3, \quad \mu_3 = \frac{1}{13}\mu_1. \qquad (A.36)$$

Notice that in this case $M_3^{sol} < M_1^{sol} + M_2^{sol}$, and the third soliton is stable in the sense that energy is required to dissociate it.

### A.5 Kink of the double sine-Gordon model as a multi-baryon

In the system (A.9)–(A.10) we perform the following reduction $\phi \equiv \varphi_1 = \varphi_2$ such that

$$\Phi_1 = \Phi_2, \ \ \Phi_3 = q\,\Phi_1, \qquad (A.37)$$

with $q$ being a real number.

Moreover, for consistency of the system of equations (A.9)–(A.10) we have

$$\mu_1 = \mu_2, \ \ \delta_1 = \delta_2 = q/2, \ \ \nu_1 = \nu_2, \ \ \nu_3 = \frac{q}{2}\nu_1, \ r = s = 1. \qquad (A.38)$$

Thus the system of eqs. (A.9)–(A.10) reduces to

$$\partial^2 \Phi_{DSG} = -\frac{\mu_1}{\nu_1}\sin(\nu_1\Phi_{DSG}) - \frac{\mu_3\delta_1}{\nu_1}\sin(q\,\nu_1\Phi_{DSG}), \quad \Phi_{DSG} \equiv \beta_0\phi. \quad (A.39)$$

This is the so-called *two-frequency sine-Gordon* model (DSG) and it has been the subject of much interest in the last decades, from the mathematical and physical points of view.

If the parameter $q$ satisfies

$$q = \frac{n}{m} \in \mathbb{Q} \quad (A.40)$$

with $m, n$ being two relative prime positive integers, then the potential $\frac{\mu_1}{\nu_1^2}(1 - \cos(\nu_1\Phi_{DSG})) + \frac{\mu_3}{2\nu_1^2}(1 - \cos(q\nu_1\Phi_{DSG}))$ associated to the model (A.39) is periodic with period

$$\frac{2\pi}{\nu_1}m = \frac{2\pi}{q\,\nu_1}n. \quad (A.41)$$

Then, as mentioned above the theory (A.39) possesses topological excitations.

From (A.2) and the set of parameters (A.38) one has the relationships

$$\Phi_1 = \Phi_2 = \frac{1}{q}\Phi_3 = \nu_1\phi. \quad (A.42)$$

And from (A.6)–(A.7) and (A.38) one gets the relationship

$$n_3 = \frac{q}{2}(n_1 + n_2). \quad (A.43)$$

So, taking into account the $QCD_2$ motivated formula (2.17) and (A.43) one computes the baryon number of this DSG solution

$$\mathcal{Q}_B^{(DSG)} = N_C\left(1 + \frac{2}{q}\right)n_3, \quad n_3 \in \mathbb{Z}, \quad (A.44)$$

where the superindex $(DSG)$ refers to the associated DSG solution.

In the following we will provide some kink solutions for a particular set of parameters. Consider

$$\nu_1 = 1/2, \ \delta_1 = \delta_2 = 1, \ \nu_2 = 1/2, \ \nu_3 = 1/2 \ \text{and} \ q = 2, n = 2, m = 1 \quad (A.45)$$

which satisfy (A.38) and (A.40). This set of parameters provide the so-called *double sine-Gordon model* (DSG), such that from (A.42) and (A.45) the field configurations satisfy

$$\Phi_1 = \Phi_2 = \frac{1}{2}\Phi_3 = \frac{1}{2}\phi. \quad (A.46)$$

Its potential $-[4\mu_1(\cos\frac{\Phi_{DSG}}{2} - 1) + 2\mu_3(\cos\Phi_{DSG} - 1)]$ has period $4\pi$ and has extrema at $\Phi_{DSG} = 2\pi p_1$, and $\Phi_{DSG} = 4\pi p_2 \pm 2\cos^{-1}[1 - |\mu_1/(2\mu_3)|]$ with $p_1, p_2 \in \mathbb{Z}$; the second extrema exists only if $|\mu_1/(2\mu_3)| < 1$. Depending on the values of the parameters $\beta_0, \mu_1, \mu_3$

the quantum field theory version of the DSG model presents a variety of physical effects, such as the decay of the false vacuum, a phase transition, confinement of the kinks and the resonance phenomenon due to unstable bound states of excited kink-antikink states (see [38] and references therein). The semi-classical spectrum of neutral particles in the DSG theory is investigated in [39].

A particular solution of (A.39) for the parameters (A.45) can be written as

$$\Phi_{DSG} := 4\arctan\left[\frac{1}{d}\ \frac{1 + h\,\exp[2\gamma(x - vt)]}{\exp[\gamma(x - vt)]}\right] \tag{A.47}$$

provided that

$$\gamma^2 = \beta_0^2\left(\mu_1 + 2\mu_3\right), \quad h = -\frac{\mu_1}{4}, \tag{A.48}$$

### A.5.1 A multi-baryon and the DSG kink ($h < 0$, $\mu_i > 0$)

For the choice of parameters $h < 0$, $\mu_i > 0$ in (A.48) the equation (A.47) provides

$$\phi = \frac{4}{\beta_0}\arctan\left[\frac{-2|h|^{1/2}}{d}\ \sinh[\gamma_K\,(x - vt) + a_0]\right], \quad \gamma_K \equiv \pm\beta_0\sqrt{\mu_1 + 2\mu_3}, \tag{A.49}$$

$$a_0 = \frac{1}{2}\ln|h|.$$

This is the DSG 1-kink solution with mass

$$M_K = \frac{16}{\beta_0^2}\gamma_K\left[1 + \frac{\mu_1}{\sqrt{2\mu_3(\mu_1 + 2\mu_3)}}\ln\left(\frac{\sqrt{\mu_1 + 2\mu_3} + \sqrt{2\mu_3}}{\sqrt{\mu_1}}\right)\right]. \tag{A.50}$$

Since one must have $\frac{\mu_3}{\mu_1} > \frac{1}{2}$ (see below for the range of possible values of these parameters) the potential supports one type of minima and thus there exists only one type of topological kink [40]. So, the DSG model possesses only the topological excitation (A.49) relevant to our QCD$_2$ discussion.

One can relate the parameters $\mu_j$ in (A.1) to the mass parameters $m_i$ in the effective lagrangian of QCD$_2$ in (2.10). So, for the "physical values" $N_f = 3$ and $e_c = 100$MeV for the coupling and taking into account (2.3), (2.5) and (2.7) one has for large $N_C$

$$\mu_j = 2\frac{m_j}{m_0}m^2 \approx N_C\,m_j\,124(\text{MeV}), \tag{A.51}$$

thus, the $\mu_j's$ have dimension $(\text{MeV})^2$.

For the values of the mass parameters $\mu_1$, $\mu_3$ in the range $[10^3\,,\,5 \times 10^4](\text{MeV})^2$ (take $m_1 \approx m_2 \approx 52\,\text{MeV}$; $m_3 = 4\,\text{MeV}$, notice that these values satisfy the relationship (A.22)) one can determine the values of the ratio $\kappa$ between the kink (A.50) and the third soliton (A.30) masses

$$\kappa \equiv \frac{M_K}{M_3^{sol}}, \quad 4 < \kappa < 4.2 \tag{A.52}$$

The baryon number of this DSG kink solution is obtained from (A.44) taking $q = 2$, $n_3 = 2$

$$\mathcal{Q}_B^{(K)} = 4 N_C, \tag{A.53}$$

where the superindex $(K)$ refers to the associated DSG kink solution.

The above relations (A.52)–(A.53) suggest that the decay of the *kink* to four solitons $\{M_j^{sol}\}$ $(j = 1, 2, 3)$ is allowed by conservation of energy and charge, however one can see from the kink dynamics that it is a stable object and its fission may require an external trigger. For similar phenomena in soliton dynamics see ref. [41].

Let us emphasize that the baryons with charges $2 n_3 N_C$ [ set $q = 2$ in (A.44)] for $n_3 = 1, 2, \ldots$ are assumed to be bound states of $2, 4, \ldots$ "basic" baryons, and so, they would correspond to di-baryon states like *deuteron* ($_1^1 H^+$) and the "$\alpha$ particle" ($_2^4 He^+$). However, we have not found, for the $QCD_2$ motivated parameter space $(\mu_1, \mu_3)$ any kink with baryon number $2 N_C$. These $2-$baryons are expected to be found in the $2-$soliton sectors of the GSG model. Notice that in our formalism the four-baryon appears already for $N_f = 3$ as a DSG kink with topological charge (A.53). In the formalism of refs. [4, 37] the multibaryons have baryon number $k N_C$ ($k \leq N_f - 1$), so the $(N_f - 1)-$baryon is the one with the greatest baryon number.

## A.6 Configuration with baryon number $3 N_C$

These solutions do not form stable configurations, nevertheless we describe them for completeness. Let us take $\varphi_1 = \varphi_2$, so one has two 1-soliton solutions $\hat{\varphi}_A$ ($A = 1, 2$) of the system (A.9)–(A.10) associated to the parameters

$$\nu_1 = 1, \; \delta_1 = 1/2, \; \nu_2 = 1, \; \delta_2 = 1/2, \; \nu_3 = 1/2, \; r = s = 1. \tag{A.54}$$

As the first 1-soliton one has

$$\varphi_1 = \varphi_2 \equiv \hat{\varphi}_1, \tag{A.55}$$

$$\hat{\varphi}_1 = \frac{4}{\beta_0} \arctan\{d \; \exp[\gamma_4 (x - vt)]\}, \tag{A.56}$$

which requires

$$d^2 = 1, \;\; 38 \gamma_4^2 = \beta_0^2 \Big( 25 \mu_1 + 13 \mu_2 + 19 \mu_3 \Big) \tag{A.57}$$

This is a sine-Gordon 1-soliton associated to both fields $\varphi_{1,2}$ in the particular case when they are equal to each other. It possesses a mass

$$M_4^{sol} = \frac{8 \gamma_4}{\beta_0^2}. \tag{A.58}$$

In view of the symmetry (A.11) we are able to write from (A.57)

$$d^2 = 1, \;\; 38 \gamma_5^2 = 25 \mu_2 + 13 \mu_1 + 19 \mu_3, \tag{A.59}$$

and then one has another 1-soliton from (A.55)–(A.56)

$$\varphi_1 = \varphi_2 \equiv \hat{\varphi}_2, \tag{A.60}$$

$$\hat{\varphi}_2 = \frac{4}{\beta_0} \arctan\{d \, \exp[\gamma_5(x - vt)]\}. \tag{A.61}$$

It possesses a mass

$$M_5^{sol} = \frac{8\gamma_5}{\beta_0^2}. \tag{A.62}$$

Similarly, from (A.2) and the set of parameters (A.54) one has the following relationships

$$\Phi_1 = \Phi_2 = \Phi_3 = \hat{\varphi}_A, \quad A = 1, 2. \tag{A.63}$$

From (A.6)–(A.7) and (A.54) one gets the relationship

$$2n_3 = n_1 + n_2. \tag{A.64}$$

So, taking into account the QCD$_2$ motivated formula (2.17) and (A.32) one computes the baryon number of this GSG solution taking $n_3 = 1$

$$\mathcal{Q}_B^{(A)} = 3N_C, \tag{A.65}$$

where the superindex $(A)$ refers to the associated $\hat{\varphi}_A$ field. Therefore, the both solutions $A = 1, 2$, have the same baryon number in the context of QCD$_2$. The individual soliton solutions (A.56) and (A.61) have, each one, a topological charge $N_C$, since they are sine-Gordon solitons. Then, the configuration (A.63) with total charge $3N_C$ is composed of three SG solitons. Therefore, by conservation of energy and topological charge arguments one has that the rest mass of the static configurations $A = 1, 2$, with baryon number $3N_C$ will be, respectively

$$M_{4,5}^{config.} \equiv 3 \, M_{4,5}^{sol}, \tag{A.66}$$

where the masses $M_{4,5}^{sol}$ are given by (A.58), (A.62).

Moreover, one can verify the following relationships

$$i) \; M_{4,5}^{config.} > M_1^{sol} + M_2^{sol}; \qquad \qquad \mu_1 \neq \mu_2, \tag{A.67}$$

$$ii) \; M_4^{config.} = M_5^{config.} > M_1^{sol} + M_2^{sol} + M_3^{sol}; \qquad \mu_1 = \mu_2, \tag{A.68}$$

where the soliton masses $M_j^{sol}$ $(j = 1, 2, 3)$ are given by (A.16), (A.23), (A.30), respectively. One observes that the configurations $A = 1, 2$, do not form bound states (bound states would be formed if the inequalities (A.67)–(A.68) are reversed), and they may decay into the "basic" set $\{M_1^{sol}, M_2^{sol}\}$ or $\{M_1^{sol}, M_2^{sol}, M_3^{sol}\}$ of solitons, such that the excess energy is transferred to the kinetic energy of the solitons.

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